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Probability, Approximate Truth, and Truthlikeness: More Ways out of the Preface Paradox

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ABSTRACT
The so-called Preface Paradox seems to show that one can rationally believe two logically incompatible propositions. We address this puzzle, relying on the notions of truthlikeness and approximate truth as studied within the post-Popperian research programme on verisimilitude. In particular, we show that adequately combining probability, approximate truth, and truthlikeness leads to an explanation of how rational belief is possible in the face of the Preface Paradox. We argue that our account is superior to other solutions of the paradox, including a recent one advanced by Hannes Leitgeb (*Analysis* 74.1).

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1. Introduction

The so called Preface Paradox is a well-known puzzle about rational belief defying philosophical analysis since its appearance in the mid-1960s. The paradox apparently shows that one can rationally believe two logically incompatible propositions. Moreover, it can be turned into a powerful argument against a most cherished view of the relations between rational belief and probability. In attempting to avoid these implications of the paradox, some authors have proposed fairly radical moves, such as rejecting the requirements of consistency or logical closure for rational belief. In a recent paper, Hannes Leitgeb [2014] has instead devised a way out that avoids such radical departures from philosophical common sense. His proposal is simple and intuitively compelling; still, as we argue, it harbours some serious defects. In this contribution, we take advantage of the basic intuition underlying Leitgeb's proposal in order to explore more adequate ways out of the Preface Paradox. To this purpose, we employ the notions of truthlikeness and approximate truth as studied within the post-Popperian research programme on verisimilitude [Niiniluoto 1987, 2011; Schurz and Weingartner 1987, 2010; Kuipers 2000; Festa 2011; Cevolani, Crupi, and Festa 2011; Oddie 2014]. In particular, we show how combining the three notions mentioned in the title—probability, truthlikeness (or verisimilitude), and approximate truth—provides an explanation of how rational belief is possible in the face of the Preface Paradox.

The outline of the paper is as follows. Section 2 presents the Preface Paradox and briefly surveys some standard reactions to it, including Leitgeb's proposal. In section 3 we introduce the notions of truthlikeness and of approximate truth, and discuss their relationships and differences. Such concepts are linked to that of probability in section 4, where two different ways out of the Preface Paradox are outlined. These are worked out in detail in sections 5 and 6, where the 'approximate truth account' and the 'truthlikeness account' of the paradox, respectively, are developed and illustrated, including basic logical results. Section 7 draws philosophical conclusions and sketches two possible extensions of the proposed approach that hint at unsolved problems and future tasks.

2. A Puzzle about Rational Belief

Adam, an academic historian, has just published his last book. This is a big work, containing a great number m of different claims—call them $b_1,\ldots,b_m$—about the relevant subject matter. Adam is a serious scholar, and has carefully checked and re-checked the contents of his book: he's thus ready to assert with great confidence that each statement b_i—and hence their conjunction b—is true. Still, having published other books in the past, Adam is perfectly conscious of his own fallibility as a writer and researcher. Thus, in the preface of his book, he acknowledges that such an ambitious and long work is bound to contain some error. But this amounts to saying that Adam believes that at least one of the claims $b_1,\ldots,b_m$—hence their conjunction b—is false. In the end, Adam seems rationally entitled to believe both b and $\neg b$—namely, to entertain logically incompatible beliefs.

The story just told is a slight variation on the one, originally presented by David Makinson [1965], that popularized the now-classical Preface Paradox. The paradox raises a puzzle for the analysis of rational belief or acceptance, since Adam seems equally well justified in believing each of b and $\neg b$. In particular, as far as the latter belief is concerned, our intuitions about the fallibility of human knowledge tell us that nobody can be fully certain of the literal truth of any factual statement. And this would explain why, in the preface, Adam avoids committing himself to the full truth of what he has published.

In view of this, a tempting way out of the paradox is to resort to the idea of credences or degrees of belief, in the form of epistemic probabilities. According to this idea, by publishing his book Adam shows that he has a high degree of belief $\mathrm{P}(b_i)$ in each of the claims made in the book, where P is an epistemic probability distribution over the statements of the relevant language. It follows that $\mathrm{P}(b)$ cannot be high, provided that the claims are not in strong mutual support of each other, since then the probability of a conjunction decreases with every conjunct added to it. For instance, suppose that $b_1,\ldots,b_m$ are probabilistically independent and that $\mathrm{P}(b_i) = 0.9$ for each of them. Then the probability of their conjunction is $\mathrm{P}(b) = 0.9^m$, which quickly tends to zero as m increases. In short, the probability calculus alone guarantees that Adam will have a low degree of belief in b and a correspondingly high degree of belief in its negation—that is, in the prefatory statement $\neg b$ (since $\mathrm{P}(\neg b) = 1 - \mathrm{P}(b)$). Thus, no paradox arises: Adam is perfectly rational in attaching high confidence to each of the claims in the book, but also to the negation of their conjunction.

As convincing as this account of the paradox may be, it is seriously unsatisfactory for at least one reason. In fact, it leaves completely unanalyzed the notion of belief or acceptance that we started with in the first place. Adam's story didn't mention quantitative credences or probabilistic degrees of belief. It was presented entirely in terms of 'qualitative' or 'yes-or-no' belief—the point being that qualitative belief or acceptance is nevertheless assumed to be *fallible* and hence to have a probability smaller than 1. In contrast, the purely probabilistic way out of the paradox presented above only works in so far as one refrains from speaking of belief or acceptance, remaining content with degrees of belief only. To make sense of Adam's full beliefs, something more is needed.

A possible, and very natural, way to go is the following. According to what is often called the *Lockean thesis*, Adam should rationally believe or accept a given statement h just in case his corresponding degree of belief $P(h)$ is sufficiently high—that is, higher than some 'acceptance threshold' σ with $0.5 \leq \sigma < 1$ [Foley 1992]. This 'high-probability' view of belief is apparently very plausible, and would provide a natural reconstruction of the qualitative notion of belief/acceptance within a quantitatively probabilistic framework. Unfortunately, this view is incoherent, or at least incompatible with other basic and widely shared intuitions concerning the notion of belief. One problem—another being the well-known Lottery Paradox, which we shall not consider in this paper—is raised by the Preface Paradox itself. In fact, suppose that we accept the Lockean thesis: Adam believes h if and only if (iff) $P(h) > \sigma$. Assume that Adam believes each of the claims in the book: he assigns a probability $P(b_i) > \sigma$ to each of them. Then, he will also believe their conjunction b: this follows from a widely held rationality requirement—the so called principle of *conjunctive closure*, according to which, if you believe two statements, you should also believe their conjunction. But, as already mentioned, $P(b)$ can be vanishingly small. Thus, either Adam believes the highly improbable statement b (against the Lockean thesis) or he doesn't believe the conjunction of what he believes (against the conjunctive closure principle).

We are then left with two radical ways out of the Preface Paradox. The first (exemplified by Jeffrey [1970]) is to give up any qualitative belief-talk and return to purely probabilistic parlance, which allows only for degrees of belief to be attributed to rational inquirers. The second (exemplified by Kyburg [1961], Foley [1992], and Christensen [2004]) is to embrace the Lockean thesis and the high-probability view of belief, abandoning the principle of logical (conjunctive) closure of belief. Both moves have found supporters in the discussion of the Preface Paradox and related puzzles.[1] In this paper, we propose a way out of the paradox that avoids such radical departures from philosophical common sense. An important motivation for our proposal came from a recent contribution by Hannes Leitgeb [2014], to be presented below.

The basic idea underlying Leitgeb's solution of the Preface Paradox is very simple. According to it, there is a considerable difference between asserting just a couple of propositions and making a great number m of different assertions. In this latter case, what is asserted is not what is really believed. In our case, what Adam really accepts by publishing his book is not that all of $b_1,\ldots,b_m$ are true, but only that the 'vast majority' of them are [ibid.: 12]. Of course, Adam can then also believe, without contradiction, that their conjunction b is false, as he makes clear in the preface. More precisely, suppose that k is a natural number smaller than m but sufficiently close to it (we may assume that $m/2 < k < m$). Then Adam accepts what Leitgeb calls the 'statistical weakening' of b—namely, the disjunction $S_k(b)$ of all possible different conjunctions of k claims out of the $b_1,\ldots,b_m$. To illustrate: if the book contains only the three claims b_1,

[1] A further possibility is that there are acceptance rules that contradict Locke's thesis by allowing the acceptance of low probability beliefs (see, e.g., Lin and Kelly [2012]).

b_2, and b_3, and k is 2, then Adam believes that

$$(b_1 \text{ and } b_2) \text{ or } (b_1 \text{ and } b_3) \text{ or } (b_2 \text{ and } b_3)$$

is true. In short, he believes that the frequency of truths among the claims made in the book is sufficiently high—that at least $(k/m) \cdot 100\%$ of them are true.

This way out of the paradox has a couple of advantages. First, it is clear that $S_k(b)$ is logically weaker than b, hence logically compatible with the prefatory statement $\neg b$. Second, Adam can have a high degree of belief in $S_k(b)$, since the probability of $S_k(b)$ can be high even if $P(b)$ is low (more on this in section 5). Finally, Adam's beliefs can be logically closed, since he can believe whatever follows from $S_k(b)$ without contradiction or without believing improbable things [ibid.: 14].

Thus, Leitgeb's solution aims to be the best of both worlds—the high-probability view of belief, on the one hand, and the principle of conjunctive closure, on the other hand. Unfortunately, his proposal also has some serious drawbacks. The most important is that, according to Leitgeb's reconstruction, Adam doesn't accept *any* of the claims made in the book. In fact, note that $S_k(b)$ is so weak a statement that none of b_1, $\ldots, b_m$ follows from it. This amounts to saying that Adam has to suspend judgment on each of those claims. This seems an extremely demanding form of modesty, even for a very cautious inquirer. In light of the preface remark, saying that Adam doesn't believe *all* of what he asserts in the book makes perfect sense; but concluding that Adam believes *nearly nothing* of what he has written is much less convincing. What is needed here is a middle way, allowing one to say that Adam accepts *most*, if not all, of the contents of his book. In the following, we show that the notions of truthlikeness and approximate truth can deliver the intuitively right kind of assessment here. As a consequence, we are led to reject Leitgeb's solution to the paradox, which, however, points in the right direction—that is, to the idea that Adam fallibly accepts b as approximately true, or truthlike.

3. Explicating Approximate Truth and Truthlikeness

In the next sections, we shall employ the two notions of truthlikeness and of approximate truth to offer a viable account of the Preface Paradox. Since they may be quite unfamiliar to the reader, and are still sometimes confused in the literature (see, for example, Niiniluoto [1998] and Cevolani and Tambolo [2013]), this section is devoted to introducing simple definitions of truthlikeness and approximate truth, starting with a small amount of formal notation and terminology, which we shall employ in the following.

To keep things simple, we shall focus on the 'conjunctive statements' of a propositional language with n atomic sentences, $a_1, \ldots, a_n$. A conjunctive statement h is a non-contradictory finite conjunction of m 'elementary' or 'basic' statements of the language (with $m \leq n$)—that is, atomic sentences or their negations. As an example, we shall assume that the statements $b_1, \ldots, b_m$ in Adam's book are elementary propositions, describing the basic features of the underlying domain, and that their conjunction b is a conjunctive statement in the sense defined here. For the sake of generality, we also consider the case of $m = 0$, corresponding to the 'tautological' conjunctive statement $\top$, making no elementary claim at all about the domain. As another extreme case, if $m = n$ then h is a so-called (propositional) *constituent* of the language—that is, a conjunction of exactly n elementary propositions, one for each atomic statement. Constituents are the strongest conjunctive statements of the language. They are also known as state descriptions, since they can be regarded as the most complete descriptions of the $q = 2^n$ alternative states of affairs (or 'possible worlds') $w_1,\ldots,w_q$ expressible within the language. The only true constituent w^* of the language is the complete true description of the actual world. In this sense, w^* can be construed as 'the (whole) truth' about the underlying domain.

The degree of truthlikeness of a proposition h expresses the closeness or similarity of h to the whole truth about the domain—that is, to the true constituent of the relevant language. When h is a conjunctive statement, we may safely assume that h is closer to the truth w^*, the more true (elementary) claims and the less false (elementary) claims that h makes. A simple measure of truthlikeness along these lines is the one proposed within the 'basic feature' approach to truthlikeness developed by Cevolani, Crupi, and Festa [2011]. This measure amounts to the difference between the normalized number t of the true claims (the 'matches') of h and the normalized number f of its false claims (the 'mistakes' of h), weighted by a parameter $\varphi > 0$ that balances the relative importance of matches and mistakes in assessing the truthlikeness of h:[2]

Definition 1. $Tr_\varphi(h) = \frac{t}{n} - \varphi\frac{f}{n}$

Borrowing some betting terminology, if $\varphi > 1$ then the 'cost of error' incurred by h for making a false elementary claim counts more than the 'gain of truth' obtained for making a true elementary claim. If $\varphi = 1$, the gain from a match is exactly the same as the cost for a mistake. Finally, if $\varphi < 1$ then the gain from a match is greater than the cost due to a mistake. Note that $\mathrm{Tr}_\varphi(h)$ varies between a maximum value of 1 (when h is the truth itself) and a minimum value of $-\varphi$ (when h is the inverse of the truth—namely, the conjunction of the negations of all true elementary statements). The truthlikeness $\mathrm{Tr}_\varphi(\top)$ of a tautology is 0. It can be taken as a natural threshold value, since $\mathrm{Tr}_\varphi(h)$ will be greater than 0 if, overall, the matches of h weigh more than its mistakes, smaller than 0 in the opposite case, and 0 when the matches and mistakes of h have equal weight.

While truthlikeness is closeness to the *whole truth*, approximate truth is closeness *to being true*. In other words, h is approximately true when it has a high *degree of truth.* For a conjunctive statement h, its degree of (approximate) truth is defined simply as the ratio $\mathrm{AT}(h)$ of the number t of its true claims to the total number m of its claims:[3]

Definition 2. $AT(h) = \frac{t}{m}$

In words, $\mathrm{AT}(h)$ is high when most of its claims are true. It follows that all true statements (for which $t = m$) have the same degree of truth—namely, the maximum, 1.

This fact makes clear what the main difference between truthlikeness and approximate truth is. $\mathrm{AT}(h)$ may well be high (or even maximal); but still $\mathrm{Tr}_\varphi(h)$ may be small, because the information content of h is low—that is, m may be too small, in comparison to n, to make h truthlike. In short, a high degree of truth doesn't guarantee a high

[2] In principle, the notation should make reference to the particular statement h under consideration. Accordingly, we should write m_h, t_h, f_h, and so on. We avoid doing this, however, since this would uselessly burden the notation.

[3] On the notion of approximate truth in general, see Niiniluoto [1987: 176–7, 218–19] and Festa [1999: 72ff]. For the special case of definition 2, see Cevolani [2014: 65–6], where AT(h) is called the 'accuracy' of h.

degree of truthlikeness. However, if $\mathrm{Tr}_\varphi(h)$ is high, then $\mathrm{AT}(h)$ has to be reasonably high as well. For instance, assuming $\varphi = 1$, if $\mathrm{Tr}_1(h) > s/n$ (for some threshold value s such that $0 \leq s < n$), then $\mathrm{AT}(h) > (s/2m) + (1/2)$.[4] In particular, if $\varphi = 1$ and h is more truthlike than a tautology (that is, if $\mathrm{Tr}_1(h) > 0$), then h makes more matches than mistakes (that is, $\mathrm{AT}(h)$ is greater than ½).

A couple of toy examples will be useful for illustrating these points. Let us consider the following conjunctive statements:

$$
\begin{aligned}
&h = a_1\\
&h' = a_1 \ \& \ a_2\\
&h'' = a_1 \ \& \ a_2 \ \& \ a_3 \ \& \ a_4 \ \& \ \neg a_5\\
&w^* = a_1 \ \& \ldots \& \ a_n
\end{aligned}
$$

with w^* representing the whole truth concerning a given domain (of course, we assume $n \geq 5$). Assuming again $\phi = 1$, it is easy to check that the above four statements are in increasing order of truthlikeness:

$$\mathrm{Tr}_1(w^*) = 1 > \mathrm{Tr}_1(h'') = 3/n > \mathrm{Tr}_1(h') = 2/n > \mathrm{Tr}_1(h) = 1/n$$

Note that h' is closer to the truth than h is, since h' makes all the true claims of h, and one more. More generally, among true statements truthlikeness increases with content: if h and h' are true, and h' entails h, then h' is more truthlike than h. Note also that a false statement like h'' can be closer to the truth than true ones like h and h'. In fact, the greater true content of h'' can outweigh, so to speak, the decrease in truthlikeness due to its mistake.

On the other hand, the degree of truth of h only measures closeness to the truth as expressed 'in a h's own language'—namely, by its own propositional variables. Therefore, true (non-tautological) statements have always a maximal degree of truth, independently of their information content. The degrees of truth of the four conjunctions in our example above are ordered as follows:

$$\mathrm{AT}(w^*) = 1 = \mathrm{AT}(h) = 1 = \mathrm{AT}(h') = 1 > \mathrm{AT}(h'') = 4/5$$

Of course, truthlikeness and approximate truth coincide for the whole truth itself: $\mathrm{Tr}_\phi(w^*) = \mathrm{AT}(w^*) = 1$; for all of the other statements, however, Tr_ϕ and AT differ in general. This hints at two different ways out of the Preface Paradox, ones that we outline in the next section and develop fully in sections 5 and 6.

4. Probability, Approximate Truth, and Truthlikeness in the Preface Paradox

As explained in section 1, Adam cannot have a high degree of belief in the truth of b (the conjunction of the claims in his book), since the probability of b is bound to be quite low. On the other hand, from our discussion in the previous section it should be clear that Adam could well believe that b is approximately true, or even truthlike. In

[4] More generally, $\mathrm{Tr}_\phi(h) > (s/n)$ iff $\mathrm{AT}(h) > (s/(1+\phi)m) + (\phi/(1+\phi))$. *Proof*: $\mathrm{Tr}_\phi(h) = (t/n) - \phi(f/n) > (s/n)$ iff $t - \phi(m-t) > s$ iff $t(1+\phi) > s + \phi m$ iff $(t/m) = \mathrm{AT}(h) > (s/(1+\phi)m) + (\phi/(1+\phi))$.

turn, this helps to make sense of the intuition underlying Leitgeb's solution of the paradox—that most, if not all, of what Adam says in the book is true. In the following, we present in more detail this new way out of the paradox.

We start by clarifying the difference between the notion of probability, on the one hand, and those of truthlikeness and approximate truth, on the other. This is a crucial distinction, that Popper had the merit of first pointing out. In his own words [1963: 237]:

> The differentiation between these two ideas [truthlikeness and probability] is the more important as they have become confused; because both are closely related to the idea of truth, and both introduce the idea of an approach to the truth by degrees.… [P]robability … represents the idea of approaching logical certainty … through a gradual diminution of informative content. Verisimilitude [i.e. truthlikeness], on the other hand, represents the idea of approaching comprehensive truth.

As Popper highlights, probability is a decreasing function of logical strength and, in this sense, of content. On the contrary, truthlikeness must be positively associated with high content: while, for instance, a_1 & a_2 cannot be more probable than a_1, the former statement is closer to the truth a_1 & … & a_n than the latter one is.

Moreover, the degree of truth of h and its probability are completely distinct notions, since an increase of content may well increase the degree of truth but not the probability of h. For instance, while a_1 & $\neg a_2$ cannot be more probable than $\neg a_2$, the former statement has a higher degree of truth (1/2) than the latter does (0).

As far as the Preface Paradox is concerned, the above illustrated distinction implies that b (the long conjunction) could be assessed as truthlike, or as approximately true, even if b is not probable. This is for a simple reason. In fact, suppose that most of the claims $b_1,\ldots,b_m$ in the book were actually true: then AT(b) would be high, and $\mathrm{Tr}_\phi(b)$ may be reasonably high too (depending on the value of n)—and in any case higher than $\mathrm{Tr}_\phi(b_i)$, for each b_i. Accordingly, if Adam has reason to think that each b_i is probably true, he may well believe that b is truthlike, or at least approximately true; moreover, his belief may be justified, in the sense that it would itself have a sufficiently high probability. To make sense of this, one needs to adequately combine the notions of probability, approximate truth, and truthlikeness. There are at least two different ways to do this: the first makes use of the notion of *probable approximate truth*, the second employs the idea of *expected truthlikeness* (see Niiniluoto [1987: ch. 7, 2011: 345–7]). The two following sections explore these possibilities.

5. The Approximate Truth Account of the Preface Paradox

Assume that an epistemic probability distribution P is defined over the possible worlds $w_1,\ldots,w_q$. Thus, $\mathrm{P}(w_i|e)$ is the degree to which a rational inquirer like Adam believes that w_i is the actual world, conditional on his total evidence e. If Adam believes that some constituent w_i is the true one, he will assess the degree of truth of h with respect to w_i. More formally, let us slightly stretch our notation and write AT(h,w_i) to denote the degree of truth of h if w_i were actually the truth. Note that, if AT(h,w_i) is high, AT (h,w_j) will be also quite high for all worlds w_j that are reasonably close or similar to w_i. More precisely, let us define the *neighbourhood* of h as the class of those worlds where the degree of truth of h is greater than some given threshold k/m (with $0 \leq k \leq m$):

Definition 3. $N_k(h) = \left\{w_i : AT(h, w_i) \geq \frac{k}{m}\right\}$

$N_k(h)$ defines itself a proposition—a sort of 'blurred' version of h, as Niiniluoto [2011: 346] puts it—that is weaker than h, in the sense that h entails $N_k(h)$ but not vice versa. More precisely, $N_k(h)$ is the proposition that the degree of truth of h is at least k/m.

We define the *probable approximate truth* of h—that is, the probability that the degree of truth of h is at least k/m—as the probability that the true constituent w^* is in the neighbourhood of h [ibid.]:

Definition 4. $PAT_k(h \mid e) = \mathrm{P}\left(AT(h) \geq \frac{k}{m} \mid e\right) = \sum_{w_i \in N_k(h)} P(w_i \mid e) = P(N_k(h) \mid e)$

$PAT_k(h \mid e)$ is typically higher than $\mathrm{P}(h|e)$, and equal to it only if $k = m$—that is, if the threshold value for approximate truth is chosen as the highest possible one. More importantly, $PAT_k(h \mid e)$ may be high even if $\mathrm{P}(h|e) = 0$—that is, if evidence e falsifies the proposition at hand. In fact, e might still indicate that a constituent close to h is the true one. This means that an inquirer can believe with high probability that h is approximately true, even if the evidence at disposal proves h false.

Coming back to the Preface Paradox, Adam can be fairly sure that b is false—an assessment reflected in the prefatory remark (that is, in a low value of $\mathrm{P}(b)$)—but he may still evaluate the probable degree of truth of b as quite high. Interestingly, this has a striking connection with the notion of the statistical weakening of b as defined by Leitgeb (see section 1):

Observation 1. $\mathrm{S}_k(b)$ and $N_k(b)$ are analytically equivalent (that is, their equivalence follows from their definitions).

The proof is straightforward. Recall that $\mathrm{S}_k(b)$ is the disjunction of all conjunctions of k among the m claims $b_1,\ldots,b_m$ in the book. Then, by definition, $\mathrm{S}_k(b)$ is true in all of those constituents agreeing with h on at least k claims. But such worlds are exactly those where the degree of truth of h is greater than or equal to k/m—namely, by Definition 3, the members of $N_k(b)$. It follows that $\mathrm{S}_k(b)$ and $N_k(b)$ individuate the same proposition. For the same reason, $\mathrm{S}_k(b)$ and $N_k(b)$ are equally probable. So, $PAT_k(b)$ equals the probability of Leitgeb's statistical weakening of h. If this probability is high, Adam believes that b is approximately true (to degree k) with the same high confidence as he accepts Leitgeb's statistical weakening $\mathrm{S}_k(b)$.

Observation 1 is interesting because it shows how Leitgeb's solution of the Preface Paradox can be reconstructed in terms of probable approximate truth. The great advantage of this reconstruction is that the epistemic attitude of 'belief-as-approximately-true' applies not to a disjunction like $\mathrm{S}_k(b)$, but to the conjunction b—that is, to the total content of the book. With reference to the toy example introduced in section 2, if $m = 3$ and $k = 2$, then, according to Leitgeb, Adam accepts as true the disjunction (b_1 and b_2) or (b_1 and b_3) or (b_2 and b_3); whereas, according to our solution, he accepts instead the conjunction b as approximately true (to degree 2/3). Thus, by replacing the epistemic attitude of 'belief-as-(strictly)-true' by the slightly weaker epistemic attitude of 'belief-as-approximately-true', we obtain a positive solution to the Preface Paradox, in so far as the total content of all beliefs is now believed as being approximately true.

Moreover, while the probability that conjunct b is plainly true is bounded to be low, the probability that b is approximately true will be fairly high when the number of conjuncts grows sufficiently large. Indeed, as we shall see in a moment, it is even higher than the acceptance threshold σ appearing in the Lockean thesis. Thus, Adam will be rationally justified in believing b to be approximately true. To see the point, consider m elementary (and mutually probabilistically independent) propositions $b_1,\ldots,b_m$, each with probability $P(b_i) = r > \sigma$. (For the sake of notational simplicity, we omit the reference to the evidence *e*.) What is the probability that an arbitrary conjunction *b* of two or more such propositions is approximately true to degree *k*? Since there are $\binom{m}{k}$ possible conjunctions of *m* propositions such that exactly *k* of them are true, the probability that *b* has degree of truth *k*/*m* is given by the standard binomial (or Bernoulli) formula:[5]

Observation 2. $P(\{w_i : AT(b, w_i) = \frac{k}{m}\}) = \binom{m}{k} r^k (1-r)^{m-k}$

To obtain the probability that the degree of truth of *b* is at least *k*/*m*, we have to sum the values of the above formula for all values of $AT(b,w_i)$ lying between *k*/*m* and *m*/*m*:

Observation 3. $PAT_k(b) =_{\text{def}} P(\{w_i : AT(b, w_i) \geq \frac{k}{m}\}) = \sum_{i=k}^{m} \binom{m}{i} r^i (1-r)^{m-i}$

Now, it is well known that, if $P\left(AT(b) = \frac{k}{m}\right)$ is considered as a probability distribution over the possible values of *k*, then $P\left(AT(b) = \frac{k}{m}\right)$ achieves its maximum when *k*/*m* is as close as possible to *r*. More precisely, by the Law of Large Numbers (LLN) we obtain this:

Theorem 1. If $b = b_1$ &…& b_m is a conjunction of *m* mutually independent beliefs, each having probability *r*, then:

1.1. (Weak LLN) For any arbitrarily small $\varepsilon > 0$, the probability that AT(*b*) differs from *r* by an amount not greater than ε converges to 1, when *m* goes to infinity.
1.2. (Strong LLN) It is certain that AT(*b*) converges to *r*, when *m* goes to infinity.

Theorem 1 implies that we can rationally believe with almost-certainty that a large conjunction of mutually independent beliefs, each of which having probability *r*, has degree of truth *r* (and thus at least *r*). This shows that probable approximate truth has very different properties than probable truth: the latter, unlike the former, does *not decrease with the logical strength of b* (under the condition of probabilistic independence). On the contrary, when the conjunction *b* becomes longer, and thus grows in content, the probability that the degree of truth of *b* is at least *r* increases, as a consequence of the laws of large numbers.

Interestingly, one can show that this happens already for reasonably short conjunctions. To make this plain, we define the following:

Definition 6. Given an acceptance threshold σ and a conjunction $b = b_1$ &…& b_m, the *believed degree of truth* of *b* is *k*/*m*, where *k* is the greatest integer such that $PAT_k(b) > \sigma$.

In other words, the believed degree of truth is the highest degree of truth that can be accepted with high probability (greater than σ). Table 1 displays the believed degrees of truth (computed by the formula in observation 3) of conjunctions of independent beliefs each having probability $r = 0.95$, relative to an acceptance threshold $\sigma = 0.9$. The believed degree of truth of single conjuncts equals 1—namely, full truth. For short conjunctions (of at least two conjuncts), their believed degree of truth may sink below the

[5] Each claim b_i has probability *r* of being true. Hence, each possible conjunction of *m* claims has probability $r^k(1-r)^{m-k}$ of containing *k* true claims (recall that the b_i are probabilistically independent). Since there are $\binom{m}{k}$ such conjunctions, the probability of picking up one of them is that which is given in observation 2.

Table 1. Believed degrees of truth (k/m) of a conjunction of m elementary independent beliefs b_i with $P(b_i) = 0.95$ and $\sigma = 0.9$.

m	1	2	3	4	5	10	20	100	1000	$\lim_{m\to\infty}$
k/m	1	1	0.66	0.75	0.8	0.9	0.9	0.92	0.94	0.95

probability of the single conjuncts and even below the acceptance threshold σ. However, as m increases, $\mathrm{PAT}_k(b)$ quickly converges to 0.95 (by Theorem 1). In this precise sense, one may say that, while belief-as-true is not closed under conjunction, belief-as-approximately-true is 'approximately closed' under conjunction: in fact, *almost all* conjunctions of probably approximately true beliefs are themselves probably approximately true.

6. The Truthlikeness Account of the Preface Paradox

The solution of the Preface Paradox just illustrated in terms of approximate truth justifies our epistemic practice of believing the conjunction b of all propositions whose probability passes a given threshold σ, because, although it is unlikely that b is plainly true, it is highly probable that b is approximately true to a high degree. Even in this case, however, b doesn't need to be probably *truthlike* to a comparably high degree. This is because, as recalled in section 3, a high degree of truthlikeness requires b to be not only approximately true, but also highly informative. For instance, assuming $\varphi = 1$, if the degree of truth of b is k/m, the truthlikeness of b is this:

$$k/n - f/n = k/n - (m-k)/n = (2k-m)/n,$$

which may well be low if m is small as compared to n (the total number of atomic propositions). Thus, even if Adam believes with high probability that b is approximately true to degree k/m, he will believe with the same probability that b is truthlike only to degree $(2k-m)/n$, which may be small as n increases.

This shows that the epistemic practice recommended by the approximate truth account—namely, to believe all elementary propositions that are sufficiently probable—cannot apply when the probable truthlikeness of our beliefs is at issue. We suggest instead that, in this case, the recommended epistemic practice is optimal in the sense that it *maximizes*, not the probability, but instead the expectation value of the truthlikeness of our beliefs—that is, their expected truthlikeness.

The degree $\mathrm{ETr}(h|e)$ of expected truthlikeness of h given evidence e is defined as follows [Niiniluoto 2011: 345]:

Definition 7. $ETr_\varphi(h \mid e) = \sum_{w_i} P(w_i) Tr_\varphi(h, w_i)$,

that is, as the sum of the degree of truthlikeness of h in all different possible worlds, each weighted by the corresponding probability given the evidence. $\mathrm{ETr}_\varphi(h|e)$ represents the inquirer's best estimation of the truthlikeness of h, as based on e, when the truth is unknown. Thus, a rational strategy for such an inquirer is to accept, in any given circumstance, that statement h which maximizes expected truthlikeness. Note that, as in the case of $PAT_k(h \mid e)$, $ETr_\varphi(h \mid e)$ can be high even if $P(h|e)$ is low or even zero, if e falsifies h.

According to the truthlikeness account, what Adam asserts by publishing the book is that b is his best attempt to approximate the truth about the domain under inquiry—in other words, that b maximizes expected truthlikeness, given his assessment of the relevant probabilities and the available evidence e. Still, as Adam makes clear in the preface of his book, b may be likely false, or even already falsified by e.

Interestingly, Adam has a simple way to maximize expected verisimilitude—by accepting the conjunction of all sufficiently probable elementary claims about the domain. More precisely, one can prove this (see the Appendix for a proof and for relevant caveats):

Theorem 2. If b is the conjunction of all and only basic statements $b_1,\ldots,b_m$ such that $\mathrm{P}(b_i|\mathrm{e}) > \sigma = \varphi/(\varphi+1)$, then $ETr_\varphi(b \mid e)$ is maximal.

So, if the antecedent of Theorem 2 is satisfied, Adam can rationally accept b as the statement with the highest expected truthlikeness at his disposal (conditional on the available evidence), even acknowledging in the preface that it is unlikely that b is plainly true.

Theorem 2 offers a straightforward motivation for the Lockean thesis: believing highly probable basic statements guarantees that their conjunction maximizes expected truthlikeness. Another nice implication of this result is that the acceptance threshold σ is directly computed in terms of the factor φ that balances the cost of error with the gain of truth: σ is $\varphi/(\varphi+1)$. This is an important result, since many accounts of rational belief lack a method to determine the relevant acceptance threshold. The present method of computing σ has the following attractive features.

(1) If $\varphi = 1$—that is, if the cost of error and gain of truth are equally balanced—then $\sigma = 1/2$ and it is rational to believe each b_i just in case it is more probable than not. This result mirrors the well known 'maximum' rule in the theory of prediction, which prescribes predicting a binary random event exactly when its probability is greater than 1/2 [Reichenbach 1938: 310ff].

(2) It is reasonable to set one's acceptance threshold higher than 1/2 iff $\varphi > 1$—that is, when the cost of error weighs more than the gain of truth. This is indeed the case in most ordinary decisions: indeed, in normal contexts σ is significantly higher than 1/2. For example, if the cost of an error counts nine times more than the gain of truth—that is, if $\varphi = 9$—then $\sigma = 9/10 = 90\%$.

(3) On the other hand, if you are offered a cheap bet on an event with a high gain, then you will accept the bet and predict the event even if it is improbable. For example, if $\varphi = 1/9$—that is, if the gain weighs nine times more than the cost—then $\sigma = (1/9)/(10/9) = (1/10) = 10\%$.

Another interesting aspect of Theorem 2 is that it agrees precisely with a well-known result concerning fair bets (see, for example, Howson and Urbach [1996: ch. 5]). Letting c and g denote, respectively, the cost and the gain associated with a bet on some event, the result states that the bet is fair when the probability of the event coincides with the so-called betting quotient—that is, with the ratio c/(c+g). In the present account, the relevant probability at issue is given by the acceptance threshold σ. Moreover, since φ expresses the balance between the cost c of elementary mistakes and the gain g for elementary matches (cf. definition 1), we plausibly have $\varphi = \mathrm{c/g}$. This gives us the following:

Observation 4. By Theorem 2, $\sigma = \varphi/(\varphi+1) = (\mathrm{c/g})/((\mathrm{c/g})+1) = (\mathrm{c/g})/((\mathrm{c+g})/\mathrm{g}) = \mathrm{c/(c+g)}$, which is the fair betting quotient.

In other words, in order to maximize the truthlikeness of one's beliefs, one has to accept a proposition just in case its probability is at least as great as the fair betting quotient relative to a wager for which the gain and cost are construed as the gain of truth and the cost of error, respectively.

Before concluding, let us address a point usefully raised by an anonymous reviewer. Definition 7 suggests applying the idea of expectation value also to approximate truth. The expected degree of truth of h is defined thus (see, for example, Festa [1999: 73]):

Definition 8. $EAT(h \mid e) = \sum_{w_i} P(w_i)AT(h, w_i)$

How does this notion relate to the one of probable approximate truth (cf. Theorem 1)? The answer is straightforward:

Observation 5. If $b = b_1$ &…& b_m is a conjunction of m mutually independent beliefs, each having probability r, then $EAT(b) = r$.

Thus, if the probability r of each b_i is high, the expected degree of truth of their conjunction is also high. The above result follows from the well-known fact that, to use the statistical jargon, the sample mean is an unbiased estimator of the population mean. In other words: since all b_i in the 'population' have probability r, a sample of m of them will contain, on average, $r \cdot m$ true statements. Thus, one can expect that the degree of truth of a conjunction of m independent basic statements equals their probability r.

7. Conclusions, and Some Possible Extensions

The Preface Paradox apparently shows that one can rationally believe two logically incompatible propositions. Earlier accounts of the paradox require us either to reject any possible qualitative conception of belief or to abandon as crucial a principle of rationality as the conjunctive closure of one's beliefs. In this paper, we have presented two different, but related, ways out of the Preface Paradox, each eschewing those radical moves. Both accounts justify the epistemic practice of believing a conjunction $b = b_1$&…&b_m of independent beliefs if and only if their probability passes a given threshold σ.

The first account is based on the idea that b may be false but approximately true—namely, that most of what b says is true. According to Theorem 1, when the probability of each b_i is greater than σ, the probability that b is approximately true is indeed very high. This probability converges to 1 when m grows large, but already for short conjunctions it surpasses the threshold σ in most cases: in this sense, belief in approximately true propositions is 'approximately closed' under conjunction.

The second account views b as one's best attempt to approach the whole truth about the target domain. According to Theorem 2, when the probability of each b_i is greater than σ, the expected truthlikeness of the conjunction b is maximized. This result also provides a method for computing the most reasonable acceptance threshold σ for any choice of the parameter φ which balances the weight of matches and mistakes in assessing truthlikeness: σ is just $\varphi/(\varphi+1)$.

Both accounts have the advantage of being close to scientific practice. It is common folklore that the typical attitude of scientists towards their beliefs or theories is not plain or strict truth, but is instead approximate truth or truthlikeness. Very much like Adam in his prefatory remark, scientists know that they are fallible: hence, they are pretty certain that the accepted corpus of scientific knowledge contains mistakes. Nevertheless, they are also convinced that accepting such a corpus is their best option for approaching the truth, conditional on the current state of evidence, about the relevant domain on inquiry. Our proposal shows how this attitude can be reasonably justified.

So far, our proposal has two important restrictions: namely, it is limited to beliefs that are (i) represented as conjunction of elementary propositions and (ii) qualitative in nature. In the remainder of this section, we briefly sketch how these restrictions can be overcome; detailed developments are left to future work.

7.1 Extension to Qualitative Beliefs of any Logical Format

Our account assumes that each elementary conjunct is an atomic statement or its negation. For this reason, it can only be used to assess the truthlikeness of conjunctive statements like *b*. Schurz and Weingartner [1987, 2010] have developed a conjunctive account of truthlikeness that can be applied to statements of all logical formats, including disjunctions and implications, by decomposing a statement into the conjunction (or set) of its so-called *relevant elements*. In propositional logic, the relevant elements of a statement *h* are identified with their non-contractible *clauses*: thus, *e* is a relevant element of *h* iff *e* is a non-repetitive disjunction of basic statements (in some alphabetical ordering) following from *h* and such that no proper sub-disjunction of *e* follows from *h*, too. Representing statements by clauses is a standard technique in computer science: every statement can be transformed into a logically equivalent and unique conjunction of relevant elements.

The measure developed in Schurz and Weingartner [2010: sec. 5] assesses the truthlikeness of *h* on the basis of the set $E_t(h)$ of the true relevant elements of *h* and of the set $E_f(h)$ of its false relevant elements. The truthlikeness of a true relevant element is thereby positive and ranges between 0 and 1, depending on the content of the clause (true basic statements have truthlikeness 1). Similarly, the truthlikeness of a false relevant element is negative and ranges between 0 and -1, depending on its content (false basic statements have truthlikeness -1). The truthlikeness of *h* is then defined, by analogy with the measure in definition 1, as follows (we have '+' instead of '−', because the degree of truthlikeness of false elements is negative):

Definition 9. $Tr_\phi(h) = \frac{1}{n}\sum_{e \in E_t(h)} Tr_\phi(e) + \frac{\phi}{n}\sum_{e \in E_f(h)} Tr_\phi(e)$

A measure of approximate truth can also be easily defined within the relevant element approach. It amounts to dividing the sum of the truthlikeness of each element of $E_t(h)$ through this sum increased by the absolute value of the truthlikeness of the false relevant elements in $E_f(h)$:

Definition 10. $AT(h) = \sum_{e \in E_t(h)} Tr_\varphi(e) \Big/ \left(\sum_{e \in E_t(h)} Tr_\varphi(e) + \sum_{e \in E_f(h)} |Tr_\varphi(e)|\right)$

Note that, if *h* is a conjunctive statement, then its relevant elements are just basic statements, and Definitions 9 and 10 reduce, respectively, to Definitions 1 and 2 in section 3.

7.2 Extension to Quantitative Beliefs

The account developed so far is restricted to qualitative beliefs or theories. Let us relax this restriction and assume that a theory *h* is a conjunction of quantitative statements

h_i of the following form:

$$h_i : X_i(a) = r_i$$

(in words: 'the value of magnitude X_i for object a is r_i'). Moreover, assume that an adequate measure App has been defined such that App(h_i) is the degree to which r_i approximates the true value r^* of magnitude X_i for object a, with App(h_i) ranging between 1 (if $r_i = r^*$) and −1 (if r_i is maximally distant from the true value). Then it is easy to define a measure of truthlikeness for conjunctive *quantitative* theories, again using definition 1 as a benchmark:

Definition 11. $Tr_\phi(h) = \frac{1}{n} \sum_{App(h_i)>0} App(h_i) + \frac{\phi}{n} \sum_{App(h_i)<0} App(h_i)$

Intuitively, h_i counts as a match when App(h_i) is positive, and as a mistake when App(h_i) is negative. A measure of approximate truth for conjunctive quantitative statements can be obtained in precise analogy to Definition 10.

We believe that the two proposed extensions of our proposal could provide an account of truthlikeness for arbitrary (non-conjunctive and quantitative) theories. Generalizing our major results (observations 4 and 5, table 1, and theorems 1 and 2) to the extended case would be technically more involved. This is so far an open problem and a research task for the future.[6,7]

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Appendix

Proof of Theorem 2. If $b_1,\ldots,b_m$ are all of, and the only, basic statements for which $P(b_i|e) > \varphi/(\varphi+1) = \sigma$, then $ETr_\varphi(b\,|\,e)$ is maximal.

We start by noting that the (expected) truthlikeness measures Tr_φ and ETr_φ are 'additive' in the sense that the (expected) truthlikeness of a conjunction b is the sum of the (expected) truthlikeness of its elementary conjuncts $b_1,\ldots,b_m$:

$$Tr_\varphi(b) = \sum\nolimits_{b\models b_i} Tr_\varphi(b_i) \tag{A1}$$

which follows immediately from definition 1 and observing that $Tr_\varphi(b_i) = \frac{1}{n}$ if b_i is true and $Tr_\varphi(b_i) = -\frac{\varphi}{n}$ if it is false; and

$$ETr_\varphi(b) = \sum\nolimits_{b\models b_i} ETr_\varphi(b_i) \tag{A2}$$

which is proven as follows. By definition 7, $ETr_\varphi(b\,|\,e) = \sum_{w_i} Tr_\varphi(b, w_i) \times P(w_i\,|\,e)$. By the additivity of Tr_φ (see A1), $ETr_\varphi(b\,|\,e) = \sum_{w_i}\sum_{b_j} Tr_\varphi(b_j, w_i) \times P(w_i\,|\,e)$, which is the same as $ETr_\varphi(b\,|\,e) = \sum_{b_j}\sum_{w_i} Tr_\varphi(b_j, w_i) \times P(w_i\,|\,e)$, which (by definition 7) equals $ETr_\varphi(b) = \sum_{b_j} ETr_\varphi(b_j)$.

Suppose, now, that x is a basic statement not already entailed by b. Then one can prove this:

$$ETr_{\varphi}(b \,\&\, x \mid e) \gtreqless ETr_{\varphi}(b \mid e) \text{ iff } P(x \mid e) \gtreqless \frac{\varphi}{\varphi+1} \tag{A3}$$

Proof. $ETr_{\varphi}(b \,\&\, x \mid e) > ETr_{\varphi}(b \mid e)$ iff, by additivity of ETr_{φ} (A2), $ETr_{\varphi}(b \mid e) + ETr_{\varphi}(x \mid e) > ETr_{\varphi}(b \mid e)$ iff $ETr_{\varphi}(x \mid e) > 0$ iff, by definition 7, $\sum_{w_i} Tr_{\varphi}(x, w_i) \times P(w_i \mid e) > 0$. If we now split the class of constituents into two subclasses—that of the w_i entailing x and that of those not entailing x (i.e. those entailing $\neg x$)—then we obtain $\sum_{w_i \models x} Tr_{\varphi}(x, w_i) \times P(w_i \mid e) + \sum_{w_i \models \neg x} Tr_{\varphi}(x, w_i) \times P(w_i \mid e) > 0$ iff (by definition 1) $\sum_{w_i \models x} \frac{1}{n} \times P(w_i \mid e) + \sum_{w_i \models \neg x} - \frac{\varphi}{n} \times P(w_i \mid e) > 0$ iff $\frac{1}{n} \times P(x \mid e) - \frac{\varphi}{n} \times P(\neg x \mid e) > 0$ iff, given $P(\neg x \mid e) = 1 - P(x \mid e)$, $\frac{1}{n} \times P(x \mid e) - \frac{\varphi}{n} + \frac{\varphi}{n} \times P(x \mid e) > 0$ iff $\frac{1+\varphi}{n} \times P(x \mid e) - \frac{\varphi}{n} > 0$ iff $P(x \mid e) > \frac{\varphi}{\varphi+1}$. The same proof applies if '$>$' is replaced by '$=$' or '$<$'.

Now let be $\sigma = \frac{\varphi}{\varphi+1}$. Note that, since $\varphi > 0$, $0 < \sigma < 1$. To prove theorem 2, we have to show that if b is the conjunction of all, and the only, basic statements for which $P(b_i|e) > \sigma$ $(= \varphi/(\varphi+1))$, then b maximizes expected truthlikeness. This amounts to showing that $ETr_{\varphi}(b \mid e)$ is greater than or equal to $ETr_{\varphi}(h \mid e)$, for any other conjunctive theory h. We distinguish three cases:

i. Suppose that h contains some conjunct b_i for which $P(b_i|e) < \sigma$. Then A3 guarantees that, by removing from h this conjunct, one obtains a conjunction that has a greater expected truthlikeness. So, $ETr_{\varphi}(h \mid e)$ is not maximal.
ii. Suppose that h contains only conjuncts b_i for which $P(b_i|e) > \sigma$. If these are all such conjuncts, then h is the same as b itself, by definition. Otherwise, by A3, adding to h any conjunct b_i for which $P(b_i|e) > \sigma$ and which is not already in h will increase the expected truthlikeness of the new conjunction, until all of the relevant conjuncts are added. So, again $ETr_{\varphi}(h \mid e)$ is not maximal.
iii. In the third possible case, b and h agree on all basic statements b_i such that $P(b_i|e) \neq \sigma$ and differ only on those such that $P(b_i|e) = \sigma$. For this case A3 implies that h has the same expected truthlikeness as b.

In sum, the expected truthlikeness of any alternative hypothesis h is never greater than that of b, which proves theorem 1, and it is equal to b only if h contains some additional basic statement b_i with $P(b_i|e) = \sigma$. This latter result shows that the conjunction b maximizing expected truthlikeness need not be unique. However, this does not imply that one has no reason to prefer b over other conjunctions h that also maximize expected truthlikeness by adding to b new conjuncts b_i with $P(b_i|e) = \sigma$. In fact, such an h will entail b, but not vice versa: it follows that b is the weakest, and hence the most probable, theory among the ones maximizing expected truthlikeness.[8] To the extent that having highly probable beliefs is valuable, b is then the best choice among its competitors.

Finally, one should note that Theorem 2, as stated, doesn't exclude the possibility that b is in fact inconsistent. More precisely, this happens if the parameter φ governing truthlikeness assessments is too low:

b is inconsistent (under some probability distribution P) iff $\varphi < 1$.

[8] More precisely, this holds for non-dogmatic probability distributions P—ones where $P(b_i)$ never assumes the extreme values 0 or 1, even conditional on the other basic statements.

This is shown as follows. For the right-to-left direction, assume $\varphi < 1$; then $\sigma < 1/2$ and there will be a probability distribution P such that $P(b_i|e) > \sigma$ and $P(\neg b_i|e) > \sigma$ for some b_i; hence, by Theorem 2, b will contain both of them and will be inconsistent. For the left-to-right direction, suppose that b is inconsistent—that is, it contains both some basic statement b_i and its negation $\neg b_i$. This means that both $P(b_i|e) > \sigma$ and $P(\neg b_i|e) > \sigma$, which entails that $\sigma < ½$ and $\varphi < 1$.

Thus, if $\varphi < 1$, aiming at maximizing expected truthlikeness may lead one to entertain contradictory beliefs. This is, by itself, a strong argument against allowing such low values of φ. In fact, allowing for the acceptance of logically false theories means that there will be cases where we exclude from the start the possibility of our theories being true, which is clearly undesirable. To avoid such cases, it is sufficient to limit the range of the possible values of φ, and to require $\varphi \geq 1$—which, as already observed in section 6, is indeed the natural choice in most scientific and ordinary contexts.